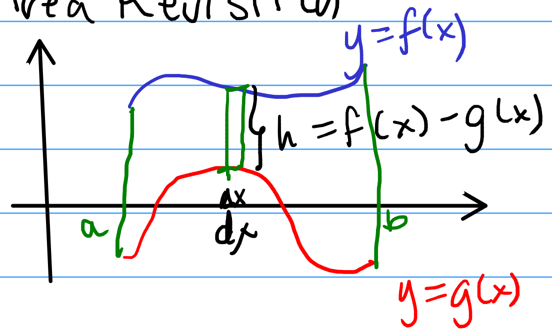
Lesson 14 - Solids of Revolution - Disks

I. Area Revisited
II. Solids of Revolution
III. Volumes of solids of Revolution

I. Area Revisited



$$\underbrace{A}_{2\text{-D}} = \int_a^b \underbrace{\left(f(x) - g(x)\right)}_{\underbrace{h}_{1\text{-D}}} dx$$

So if we want to find $\overbrace{\text{volume}}^{3\text{-D}}$, we will integrate $\underbrace{\text{area}}_{2\text{-D}}$.

II. Solids of Revolution

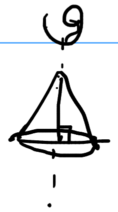
(D) A solid of revolution is a 3-D shape obtained by rotating a 2-D shape around an axis.

$\boxed{\text{EX}}$ Describe the shape of the solid of revolution

a) A rectangle rotated about the line determined by one of its sides

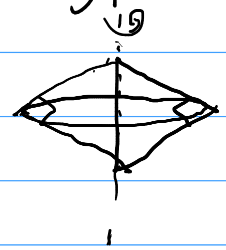
cylinder

b) A right Δ rotated about line determined by one of its legs.



cone

c) A right Δ rotated about line determined by its hypotenuse.



Cross-section - A 2-D shape that lies on a plane that crosses through a 3-D shape.

✱ In a solid of revolution, the radius of of the circular cross-sections (plane is ⊥ to axis of rotation) is ALWAYS ⊥ to axis of rotation

Name:____________________________________

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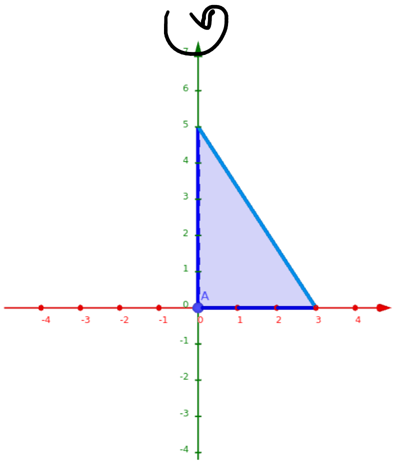
Math 16020

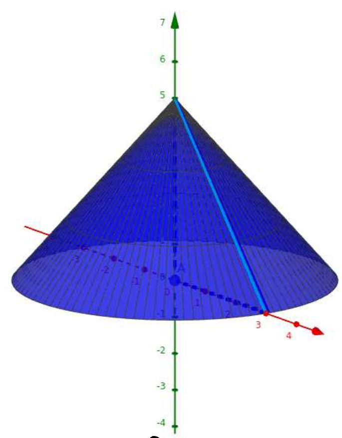
III Volume

Example 1. Find the volume of the solid that results by revolvig the region enclosed by the curves

$$x = 3 - \frac{3y}{5}, \qquad \underbrace{y = 0}_{x\text{-axis}}, \qquad \text{and } \underbrace{x = 0}_{y\text{-axis}}$$

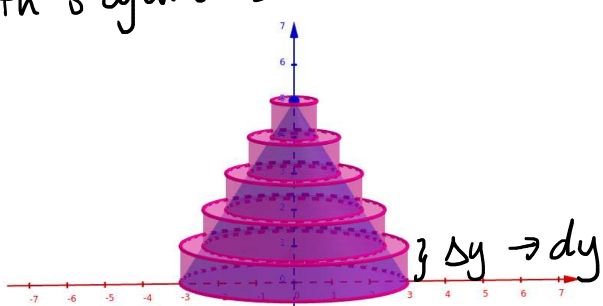
around the y-axis.





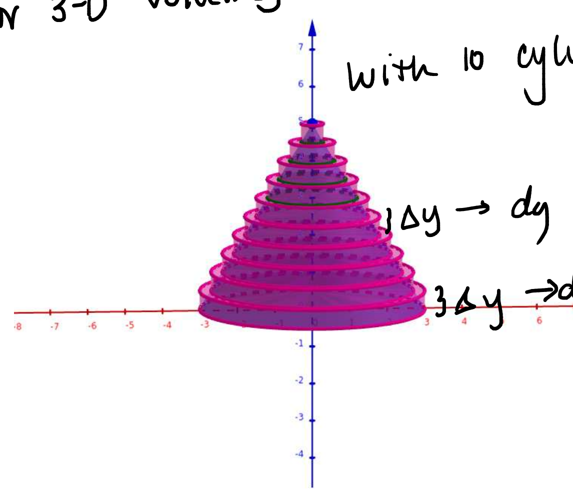
rectangles for 2-D areas are to cylinders for 3-D volumes

with 5 cylinders



$\Delta y \to dy$

with 10 cylinder



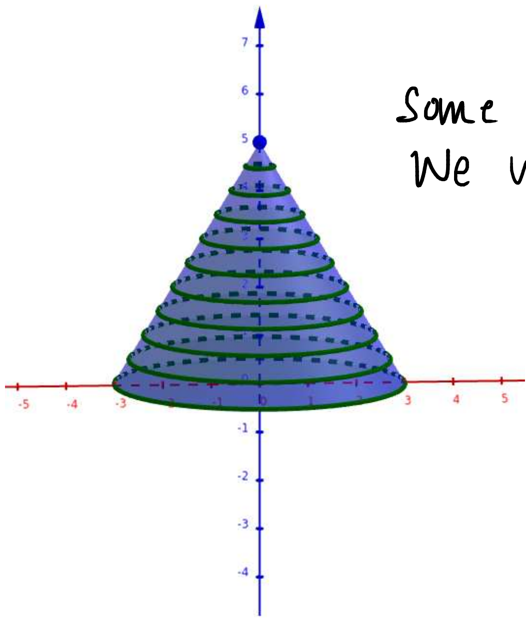
$\Delta y \to dy$

$\Delta y \to dy$

$V_{cylinder} = A_{base} \cdot \text{height}$

$= \pi r^2 \cdot \text{height}$

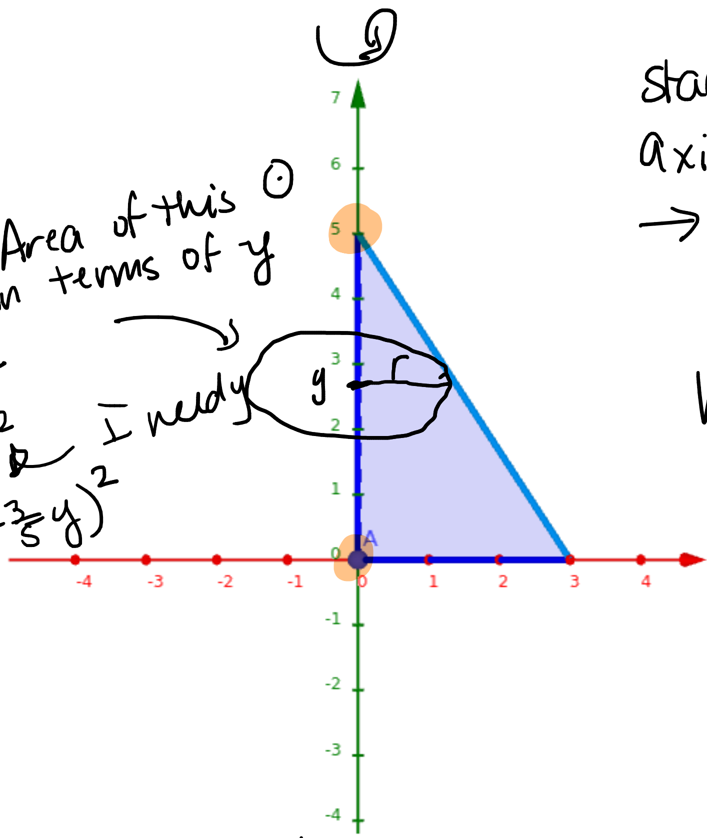
$= \pi r^2 \cdot \Delta y$



Some circular bases of the cylinders
We will integrate their areas

Stacked up cylinders along axis of rotation (y-axis)
$\rightarrow$ Δy or dy

Area of this $\odot$ in terms of y



πr^2

πx^2 I need y

$\pi\left(3-\frac{3}{5}y\right)^2$

$$V = \int_0^5 \pi\left(3 - \frac{3}{5}y\right)^2 dy$$

$$= \pi \int_0^5 \left(9 - \frac{18}{5}y + \frac{9}{25}y^2\right) dy$$

$$= \pi\left[9y - \frac{\cancel{18}\,9}{5}\frac{y^2}{\cancel{2}} + \frac{\cancel{9}\,3}{25}\frac{y^3}{\cancel{3}}\right]_0^5$$

$$= \pi\left[(45 - 45 + 15) - (0)\right]$$

$$= 15\pi$$

Euclidean geometry

$$V_\Delta = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3)^2 \cdot 5$$
$$= 15\pi$$

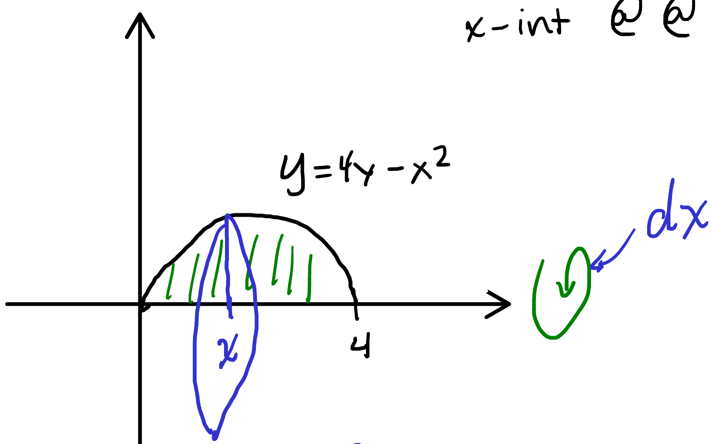
Example 2. Find the volume of the solid that results by revolvig the region enclosed by the curves

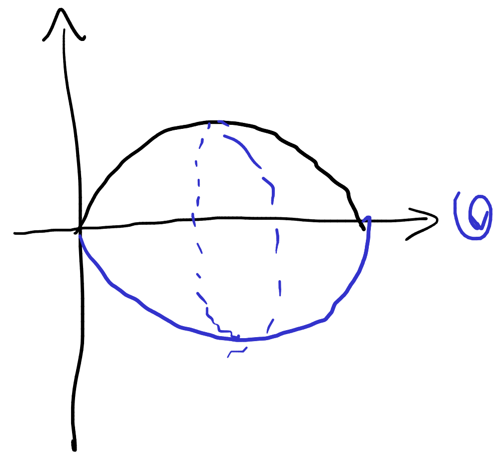
$$y = 4x - x^2 \text{and the } x\text{-axis}$$

around the x-axis.

$y = x(4-x)$

x-int @ @ $x=0$, $x=4$.





$$A_{\odot} = \pi r^2$$
$$= \pi y^2$$
$$= \pi(4x - x^2)^2$$

$$V = \int_0^4 \pi(4x - x^2)^2 \, dx$$

b/c x-axis rotation

$$= \pi \int_0^4 (16x^2 - 8x^3 + x^4)\, dx$$

$$= \pi \left[\frac{16x^3}{3} - \frac{\cancel{8}^{2} x^4}{\cancel{4}} + \frac{x^5}{5} \Big|_0^4 \right]$$

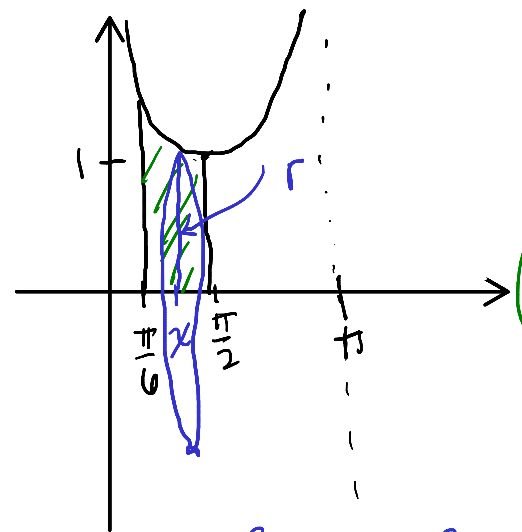
$$= \pi \left(\frac{512}{15} \right)$$

Example 3. Set up an integral to find the volume of the solid. **DO NOT** evaluate the integral.

(a) The region enclosed by the curves $y = \csc(x)$, $\underbrace{x = \frac{\pi}{6},\ x = \frac{\pi}{2}}_{\text{vertical lines}}$, and $\underbrace{y = 0}_{x\text{-axis}}$ is revolved around the x-axis.

$y = \csc(x)$

$y = \frac{1}{\sin(x)}$

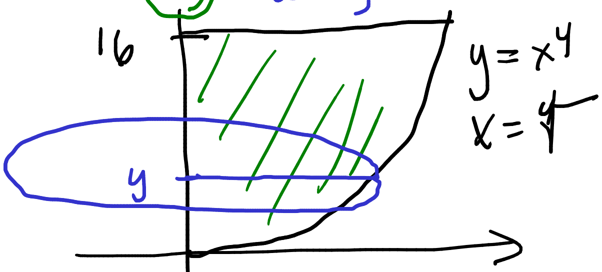


use dx

$$V = \int_{\pi/6}^{\pi/2} \pi \csc^2(x) \, dx$$

$$A_{\odot} = \pi r^2 = \pi y^2 = \pi(\csc(x))^2$$

(b) The region *in the first quadrant* enclosed by the curves $\underbrace{y = x^4}_{x=\sqrt[4]{y}}$, $y = 16$, and $\underbrace{x = 0}_{y\text{-axis}}$ is revolved around the y-axis. Use dy

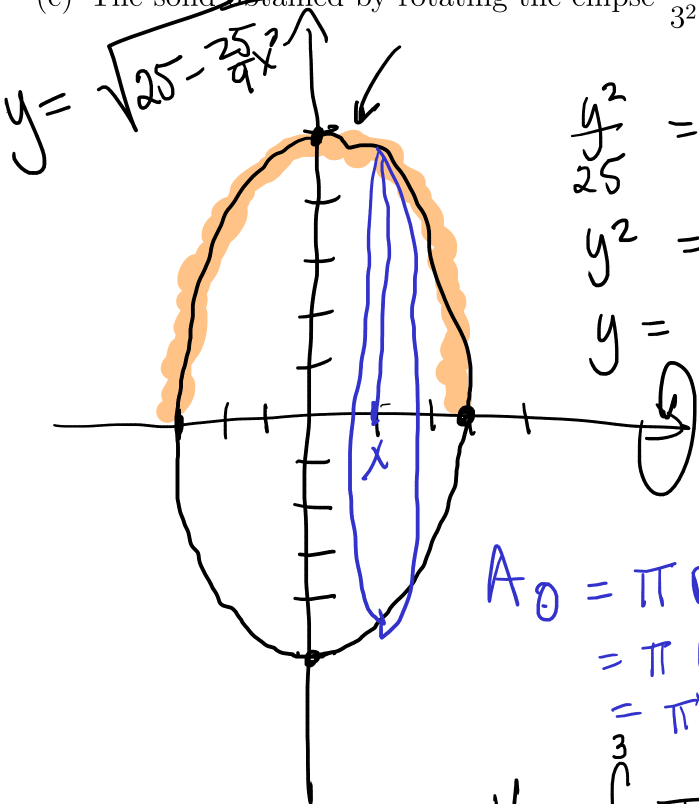


$$A_{\odot} = \pi r^2 = \pi x^2 = \pi(\sqrt[4]{y})^2$$

$$V = \int_0^{16} \pi (\sqrt[4]{y})^2 \, dy$$

$$= \int_0^{16} \pi y^{1/2} \, dy$$

(c) The solid obtained by rotating the ellipse $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ about the x-axis.



$$\frac{y^2}{25} = 1 - \frac{x^2}{9}$$

$$y^2 = 25 - \frac{25x^2}{9}$$

$$y = \pm\sqrt{25 - \frac{25}{9}x^2}$$

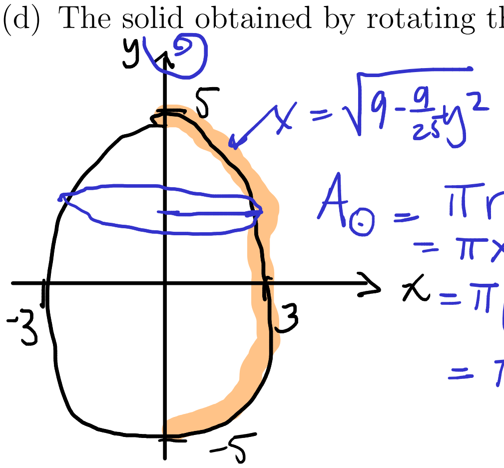
→ use dx

$$A_{\odot} = \pi r^2$$
$$= \pi y^2$$
$$= \pi\left(\sqrt{25 - \frac{25}{9}x^2}\right)^2$$

$$V = \int_{-3}^{3} \pi\left(25 - \frac{25}{9}x^2\right) \, dx$$

(d) The solid obtained by rotating the ellipse $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ about the y-axis.

$$x = \pm\sqrt{9 - \frac{9}{25}y^2}$$



$$A_{\odot} = \pi r^2$$
$$= \pi x^2$$
$$= \pi\left(\sqrt{9 - \frac{9}{25}y^2}\right)^2$$
$$= \pi\left(9 - \frac{9}{25}y^2\right)$$

$$V = \int_{-5}^{5} \pi\left(9 - \frac{9}{25}y^2\right) \, dy$$

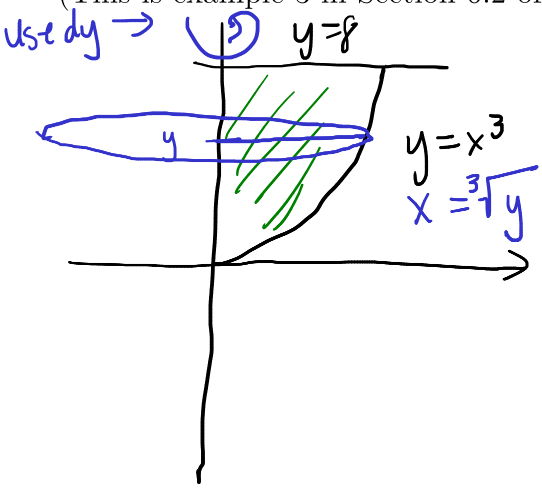
III.1 You try it!

Problem 4. Find the volume of the solid that results by revolvig the region enclosed by the curves

$$y = x^3, \quad y = 8, \quad \text{and } x = 0$$

around the y-axis. **ANSWER:** $\dfrac{96\pi}{5}$.

(This is example 3 in Section 6.2 of Stewart's 8th edition *Calculus: Early Trascendentals.*)



$$A_{\odot} = \pi r^2 = \pi x^2 = \pi\left(\sqrt[3]{y}\right)^2 = \pi y^{2/3}$$

$$V = \int_0^8 \pi y^{2/3}\, dy$$

$$= \pi \frac{3}{5} y^{5/3} \Big|_0^8$$

$$= \frac{3\pi}{5}\left[8^{5/3} - 0\right]$$

$$= \frac{3\pi}{5}(32) = \frac{96\pi}{5}$$